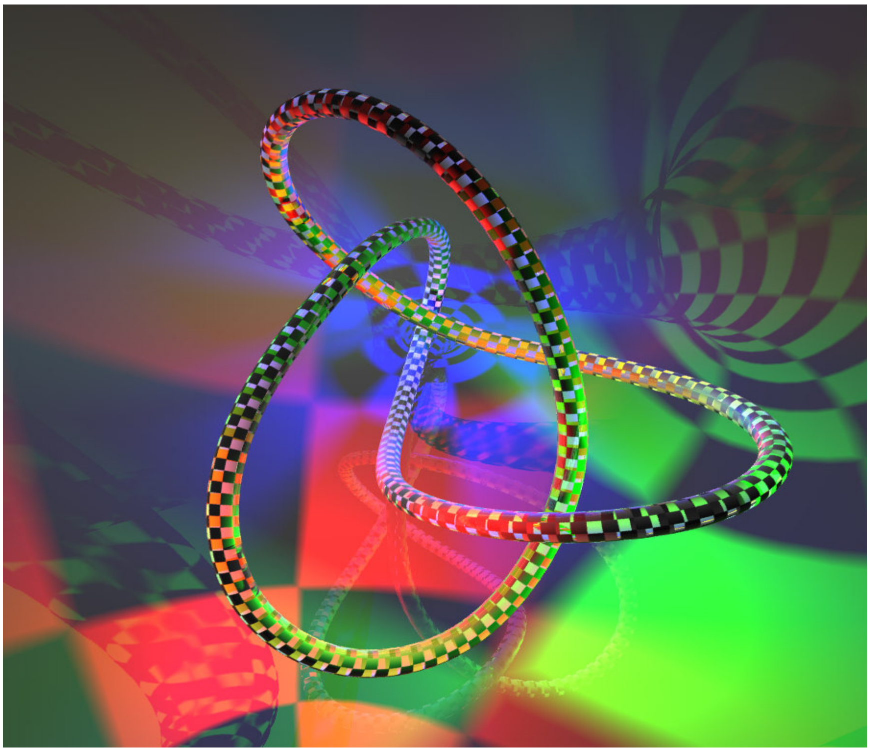
Detecting the Chirality of Knots



A Research Project in Higher Level Mathematics

Nicholas Korpelainen 2003

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- Abstract

Detecting the Chirality of Knots

Knot theory is a subfield of topology that deals with *knots*, particular embeddings of $\mathbf{S}^1$ in either $\mathbf{R}^3$ or $\mathbf{S}^3$. Distinguishing between knots is a major problem in topology: it is often useful to find *invariant* properties of knots that remain unchanged under deformations. In the case of our investigation, we look at elegant invariants for determining whether a given knot is topologically distinct from its own mirror image: *achiral* knots are deformable to their mirror images, whereas the majority of knots can be seen as pairs of distinct *chiral* knots, mirror images of each other. The following research question is asked: "How efficiently and by what means can one determine whether a knot is chiral?" As far as chiral knots are concerned, we focus on proving the chirality of those that can be drawn with less than 11 crossings. Additionally, we discuss some techniques of showing that certain types of knots must be achiral.

After introducing necessary concepts of knot theory, the chirality-detecting use of one of the most effective knot invariants, the HOMFLY polynomial, is demonstrated. The limitations of using this invariant are then commented on. A proof that the compositions of achiral knots are achiral is provided, followed by a brief investigation into the symmetries of achiral knot projections. The research into detectors of chirality is concluded by introducing mathematical methods of detecting chirality in cases where the HOMFLY polynomial is of limited use.

The problem of detecting chirality is a difficult one, but there are a few efficient invariants for chirality, which are best used together. The techniques introduced in the essay were capable of proving the chirality of all chiral knot pairs when limited to crossing numbers less than 11. A systematic method of determining whether an arbitrary knot is chiral still remains undiscovered.

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1. Introduction

1.1 Knot Theory and Chirality

Famous mathematicians such as Carl Friedrich Gauss, Peter Guthrie Tait and Lord Kelvin[1] were the first to find interest in the exploration of knot theory in the late 19th century[2]. Although this extraordinary subfield of topology[3] has been known for more than a hundred years, many of its most prominent discoveries have been made during the last few decades: scientists have found applications of knot theory useful in a wide variety of research. The study of knotted DNA-molecules and the mathematical connections between knots and the complex equations of quantum field theory are only a few of the numerous fields of science upon which knot theory has caused an immense impact.

What do we understand by a mathematical knot? A *knot* has traditionally been defined as a continuous simple closed curve in the Euclidean space $\mathbf{R}^3$. Knots are usually understood by their planar visual representations called *projections*, where “gaps” indicate the points at which the curve crosses under itself (see *figure 1)*. Following the curve in one direction, each portion of the knot that starts underneath a crossing and ends at the next under-crossing is called a *strand*. A knot such as the one below is called an *alternating knot* because it has at least one projection where the curve “alternates” between over- and under-crossings when followed in one direction. Many of the theories for knots work in the case of alternating knots, but are much harder to prove for non-alternating ones.

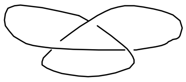

Figure 1: A projection of the left-hand trefoil knot (Alexander and Briggs notation: 3_1)[4]

The trefoil knot in *figure 1* is one of two distinct trefoil knots: the left-hand trefoil and the right-hand trefoil (see *figure 2*).

Figure 2: The two distinct, chiral trefoil knots (with a left-hand trefoil on the left).

The knot obtained by applying a crossing change[5] at each crossing of a knot’s projection is called a *mirror image* or *obverse* of the original knot. Since we have techniques to show that the two knots are distinct, we say that they are not *ambient isotopic*[6] to each other and therefore the two knots are *chiral*.

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[1] Lord Kelvin’s theory of atoms as knotted rings had a profound effect on the general interest in knot theory [*The Relation of Chemistry to Knot Theory*, URL: www.math.vt.edu/people/linnell/4994/knot.pdf]

[2] Page 31 of Colin C. Adams: *The Knot Book*, W.H.Freeman, 2001

[3] The study of the properties of geometric objects that are preserved under deformations. [Adams]

[4] This notation will be used throughout the essay. The first integer indicates the least number of crossings in any projection of the knot (the *crossing number*), whereas the index distinguishes between knots with same crossing number.

[5] Passing the over-strand (originally above in the crossing) through the under-strand (originally under the over-strand).

[6] Ambient isotopic knots are equivalent through deformations in 3D-space. Page 12 from [Adams].

Chiral knots are **not** equivalent to their mirror images, whereas *achiral*[7] knots are: they can be deformed to their mirror images. This essay deals with the research question: 'How efficiently and by what means can one determine whether a knot is chiral?" In this essay we will consider the most elegant ways of showing the chirality of all prime chiral knots with less than 11 crossings and look at some of the special properties of achiral knots. The chirality of knots has direct applications in determining whether certain knotted molecules are *stereoisomers* of one another[8]: a molecule with the shape of a chiral knot **must** be chemically chiral[9], whereas a molecule with the shape of an achiral knot might nevertheless be chemically chiral due to rigidness of bonds, bond length etc.[10]

1.2 Knot Equivalence: Reidemeister Moves

Much of knot theory is about distinguishing between different knots. Two knots are equivalent if and only if their *planar isotopies*[11] are related to each other through a sequence of equivalence moves, namely the three Reidemeister moves (see *figure 3*). Although very important for certain purposes, Kurt Reidemeister's theory has some limitations, as it does not specify the number of moves necessary to deform one projection into another. Furthermore, it does not in itself allow us to *prove* that two projections represent different knots.

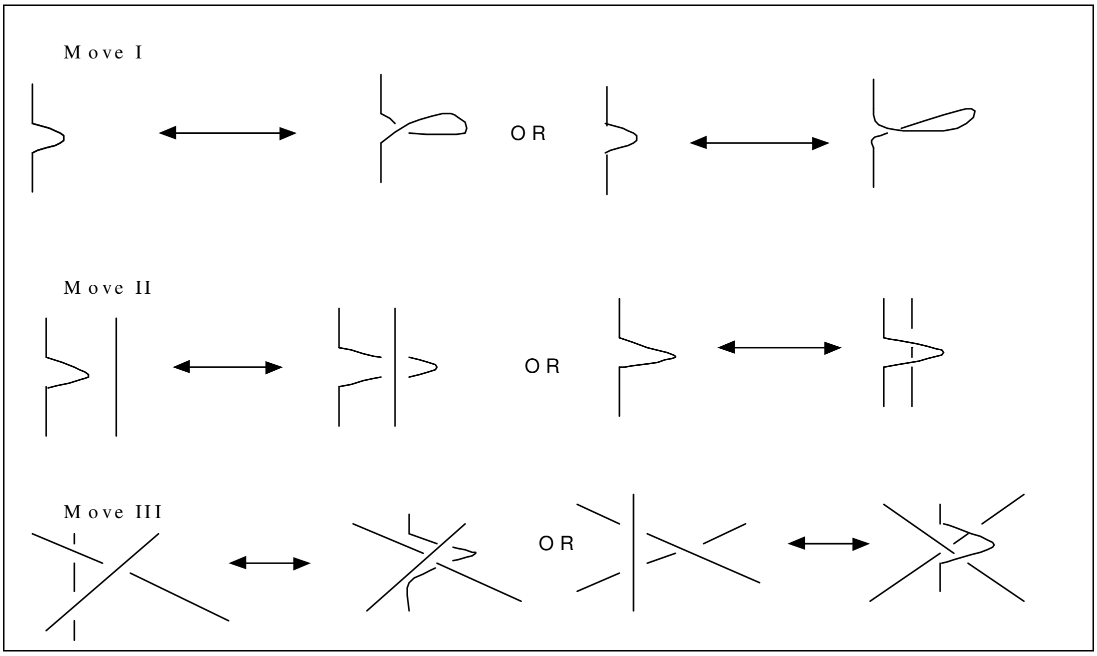

Figure 3: The three Reidemeister moves.

For a property of knots to be an *invariant*, it must remain unchanged under the Reidemeister moves. Invariants can be thought of as functions of knots – they yield the same value for any projection of a given knot. A *complete invariant* has not been found: there exists no invariant that could be thought of as a one-to-one function of knots. This means that we do not yet have a means of distinguishing between all distinct knots.

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[7] The term *amphicheiral* is also often used, especially by knot theorists in the past.

[8] *The Relation of Chemistry to Knot Theory*, URL: www.math.vt.edu/people/linnell/4994/knot.pdf

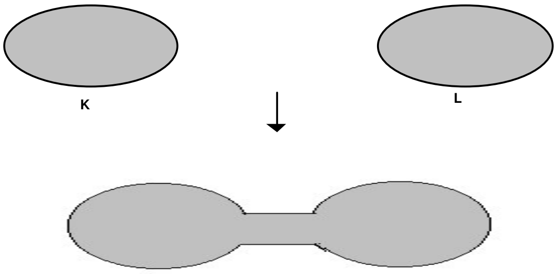
[9] In this case there exists two distinct molecules with potentially different properties, stereoisomers of one another.

[10] Page 202 of [Adams: *The Knot Book*]

[11] Planar isotopies are equivalent projections that do not differ in the configuration of crossings, but whose strands might have different deformations (as if they were highly deformable rubber) [page 12 of Adams: *The Knot Book*]

1.3 Knot Addition: Composite Knots

The operation of removing arcs from exterior strands of two nontrivial[12] knots and joining the ends (as in *figure 4*) is defined as forming a composition[13] **K#L** of the factor knots **K** and **L.** Consequently, all knots can be categorized as either *prime knots* or *composite knots*, where a prime knot is not the composition of any two other knots. As an analogy to finding the prime factors of an integer, one can always find the prime factor knots of a composite knot.



K#L

Figure 4: Forming a Composite Knot[14]

It has been shown that the *crossing number*, the least amount of crossings in any projection of a given knot, is an *additive function*[15] for certain types of knots, where forming compositions is defined as the addition of knots. For the crossing number $c(\mathbf{L\#K})$ of the composite knot **L#K**, where **L** and **K** are, for example, alternating knots, $c(\mathbf{L\#K}) = c(\mathbf{L}) + c(\mathbf{K})$.[16]

No *alternating* achiral knot can have an odd crossing number. In *table 1*, we can see that there is only one achiral (non-alternating) prime knot with odd crossing number below 16. Before 1998, knot theorists had believed in Tait's conjecture from 1890 that all achiral knots have even crossing number [17].

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Crossing number	3	4	5	6	7	8	9	10	11	12	13	14	15
Number of prime knots (excluding mirror images)	1	1	2	3	7	21	49	165	552	2176	9988	46972	253293
Number of prime knots (including mirror images)	2	1	4	5	14	37	98	317	1104	4294	19976	93670	506585
Number of achiral prime knots	0	1	0	1	0	5	0	13	0	58	0	274	1

Table 1: Statistics of Prime Knots Through 15 Crossings[18]

1.4 The Conway Notation

The Conway notation for knots is named after its creator, the English mathematician John Horton Conway. This notation is based on the idea of *tangles*. A tangle is defined as a region in the projection of a knot, where surrounding this region by a circle results in the knot intersecting with the circle exactly four times[19] (see *figure 5*). The four ends of the tangle are denoted by NW, SW, NE and SE as shown.

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[12] Nontrivial knots are defined as all knots that are distinct from the *unknot* , a knot represented by an "unknotted" circle.

[13] Known as the *connected sum.*

[14] This is a schematic diagram without actual projections – the knots **K** and **L** are **not** allowed to be trivial.

[15] An additive function f is such that $f(x+y)=f(x)+f(y)$ for defined values of $f(x)$ and $f(y)$. [David Nelson: *The Penguin Dictionary of Mathematics*, Penguin Books,1998]

[16] Proved by William Menasco in 1983. [Adams]

[17] The achiral knot 15_{224980} was found in Hoste, Thistlewaite, Weeks: *The First 1,701,936 Knots,* Math. Intell. Vol.20, No. 4, 33-48.

[18] Statistics obtained from page 33 of [Adams] and Eric W.Weisstein: *Amphichiral Knot*, Wolfram Research Inc., URL: http://mathworld.wolfram.com/AmphichiralKnot.html

[19] Page 41 of [Adams]

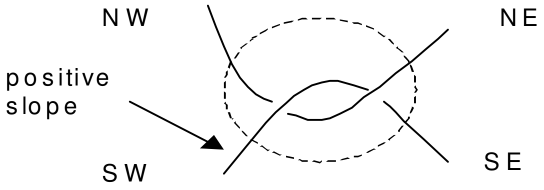

Figure 5: The tangle 2.

Let us wind two horizontal strands so that, at each crossing, the over-strand has a positive slope (see *figure 5)*. This tangle is called the tangle 2 since it has 2 ‘twists[20]’. A similar tangle with a negatively sloped over-strand at each crossing would be tangle -2. The *tangle product* 2 –4 is obtained by first reflecting tangle 2 through the NW-SE diagonal line[21] and then attaching tangle –4 to its right (see *figure 6*). The tangle 2 –4 consists of two vertical twists with positively sloped over-strands, followed by 4 horizontal twists with negatively sloped over-strands. We can add arbitrarily many integers to the end of our notation in a similar fashion: each time we want to add integer *x* to the end of the notation, we take the entire tangle, reflect it through the NW-SE diagonal line, and add *x* twists to the right. A tangle constructed by this method is defined as *rational.*

For a rational tangle with notation $a_0 a_1 a_2 ... a_n$, two notations represent the same tangle iff[22] their corresponding continued fractions of the form $a_n + \cfrac{1}{a_{n-1} + \cfrac{1}{a_{n-2} \cdots \cfrac{1}{a_1 + \cfrac{1}{a_0}}}}$ yield the same value. For a rational tangle, connecting ends NW with NE and SW with SE yields a knot that is defined as a *rational knot*, denoted similarly as the corresponding tangle. A knot obtained by connecting the ends of a tangle in this manner is called a *nominator closure*, whereas connecting NW with SW and NE with SE yields a knot that is called a *denominator closure* of the tangle.

For the *addition* of tangles **K** and **L**, the tangle **L** is simply attached to the right of **K** to obtain **K+L**.

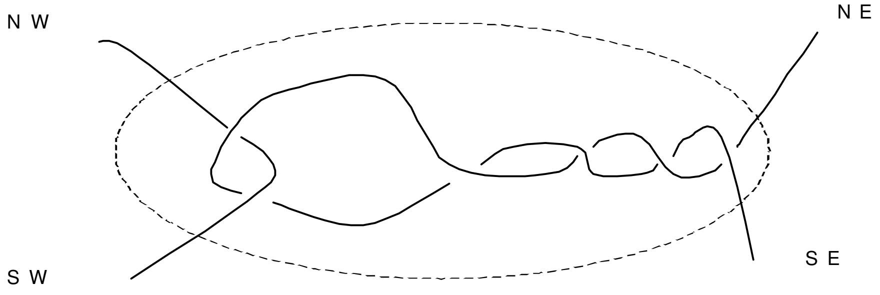

Figure 6: The tangle 2 –4.

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[20] These are actually half-twists in a mathematical sense, but for simplicity the term *twist* will be used in this essay.

[21] Imagine swapping ends SW and NE in such a way that one conserves the amount of twists.

[22] if and only if

2. Detectors of Chirality

2.1 The HOMFLY Polynomial

The HOMFLY polynomial $P_{\mathbf{K}}(l,m)$ is a two-variable Laurent polynomial and one of the most efficient invariants[23] of knots.

The HOMFLY polynomial of a knot can be calculated iteratively by applying crossing changes in an oriented[24] knot projection. This process is fully described by only two rules referring to a *skein relation* (see *figure 7*). The skein relation describes how the projections of three oriented *links*[25] differ at one chosen crossing.

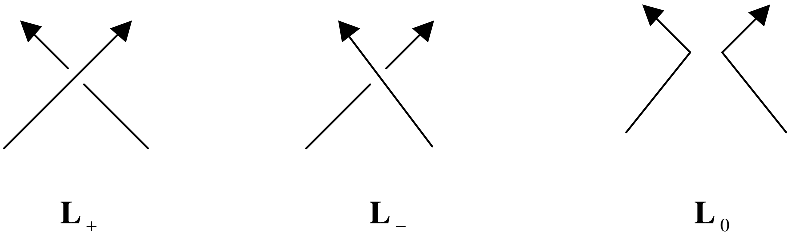

Figure 7: The skein relation for the HOMFLY polynomial[26]

Usually one chooses to define the projection for which the polynomial is calculated to be either $\mathbf{L}_+$ or $\mathbf{L}_-$, which is decided by the orientation at the chosen crossing.

The polynomials of the three links are then related by the equation:

$$\boxed{lP(\mathbf{L}_+) + l^{-1}P(\mathbf{L}_-) + mP(\mathbf{L_0}) = 0} \quad \textit{(rule 1)}$$

The second rule specifies that *the polynomial for an unknotted circle is equal to 1. (rule 2)*

Using these two rules repeatedly one can reduce the polynomial of any knot to a form containing only terms of *l* and *m*. It is enough to draw a *resolving tree* for an oriented projection of the knot (see *figure 8*). The resolving tree is a diagram that shows how the link projections involved in the calculation of the HOMFLY polynomial are related by the skein relation. One should start calculating polynomials from the bottom parts of the resolving tree – in other words by looking at the polynomials of those links which are directly related to the unknot with polynomial of 1.

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[23] It can distinguish very well between different knots. The HOMFLY polynomial contains the information of two other widely recognised and useful, but less efficient knot polynomials: the Alexander and Jones polynomials.

[24] An oriented projection is one where a sense of direction has been chosen for the knot. In the projection, this direction is then denoted by arrows that are consistent with the chosen direction. The choice of orientation does not affect the calculated polynomial in the case of knots (that is one-component links [see next footnote]).

[25] A link is similar to a knot, except that it may have more than one component, for example two linked circles.

[26] As on page 155 of [Adams]

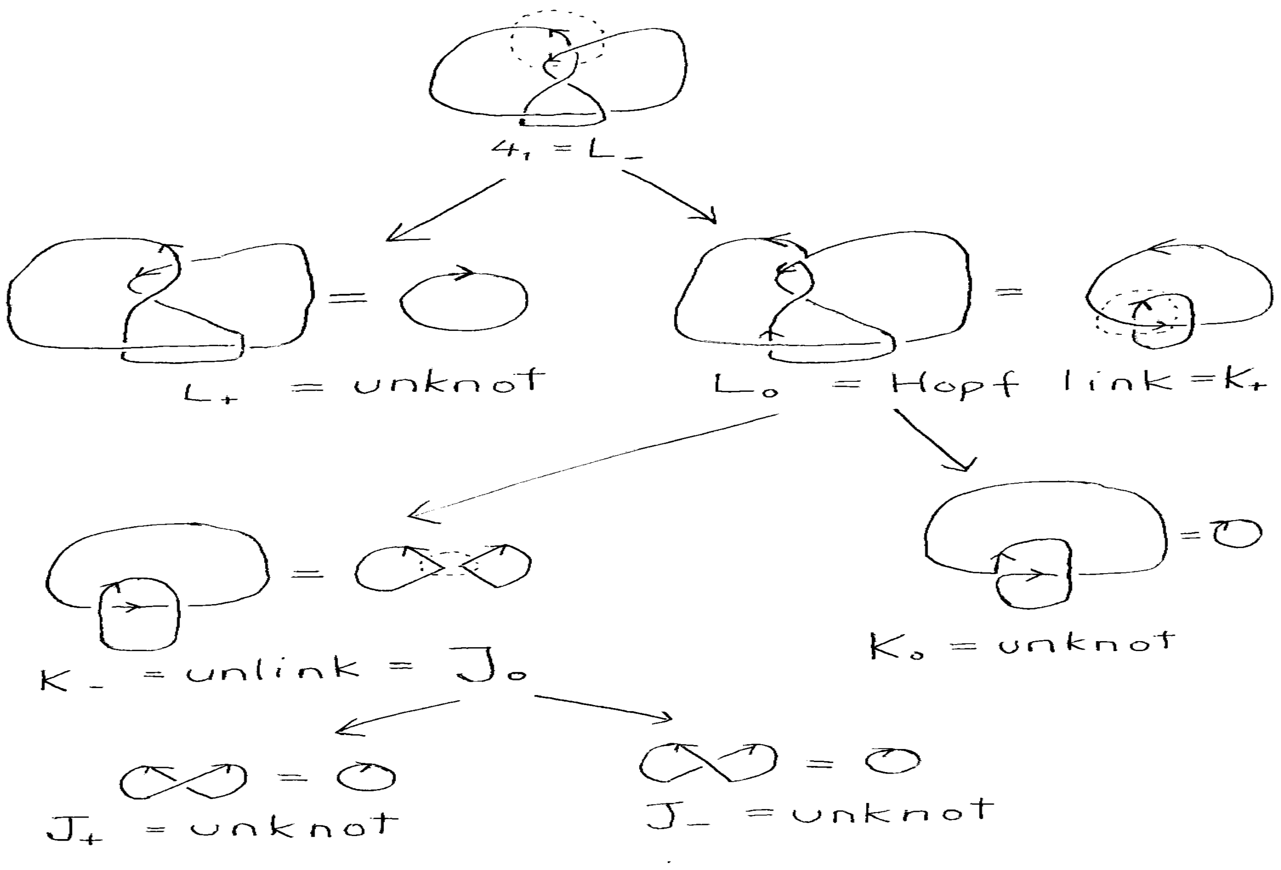

Figure 8: A resolving tree for the achiral knot 4_1 .

Using the resolving tree in *figure 8*, let us calculate the HOMFLY polynomial of the achiral prime knot 4_1:

$$P(\mathbf{J}_+) = P(\mathbf{J}_-) = P(\mathbf{L}_+) = P(\mathbf{K}_0) = 1$$

$$lP(\mathbf{J}_+) + l^{-1}P(\mathbf{J}_-) + mP(\mathbf{J}_0) = 0 \Rightarrow l + l^{-1} + mP(\mathbf{J}_0) = 0$$

$$P(\mathbf{K}_-) = P(\mathbf{J}_0) = -(l + l^{-1})m^{-1}$$

$$lP(\mathbf{K}_+) + l^{-1}P(\mathbf{K}_-) + mP(\mathbf{K}_0) = 0 \Rightarrow lP(\mathbf{K}_+) - l^{-1}(l + l^{-1})m^{-1} + m = 0$$

$$P(\mathbf{L}_0) = P(\mathbf{K}_+) = l^{-2}(l + l^{-1})m^{-1} - l^{-1}m$$

$$lP(\mathbf{L}_+) + l^{-1}P(\mathbf{L}_-) + mP(\mathbf{L}_0) = 0 \Rightarrow l + l^{-1}P(\mathbf{L}_-) + l^{-2}(l + l^{-1}) - l^{-1}m^2 = 0$$

$$P(4_1) = P(\mathbf{L}_-) = -l^2 - l^{-1}(l + l^{-1}) + m^2 = -l^2 - l^{-2} + m^2 - 1$$

If the knot **K** has HOMFLY polynomial $P_{\mathbf{K}}(l, m)$, the mirror image $\mathbf{K}^*$ will have the polynomial $P_{\mathbf{K}}(l^{-1}, m)$, the polynomial obtained by replacing the variable *l* by its inverse throughout the original polynomial[27]. This means that if a knot is achiral, that is to say that its projection is equivalent to its mirror image, the mirror image **must** yield the same polynomial. As a consequence, all achiral knots have what we call a *self-conjugate* polynomial: a polynomial where, for each term $a \cdot l^b \cdot m^c$, there exists a second

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[27] For the polynomial of the mirror image, the roles of *l* and l^{-1} are reversed, since $\mathbf{L}_+ \Rightarrow \mathbf{L}_-$ for each link.

term $a \cdot l^{-b} \cdot m^{c}$, given that *a*, *b* and *c* are integers and $b \neq 0$. If only achiral knots had self-conjugate polynomials, we could distinguish all achiral knots from chiral ones. Unfortunately, this is not the case: for example, the prime chiral non-alternating knot 9_{42} has the self-conjugate polynomial $P(9_{42}) = -2l^{-2} - 3 - 2l^{2} + (l^{-2} + 4 + l^{2})m^{2} - m^{4}$.[28]

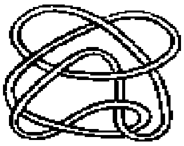

Figure 9: The chiral prime knot 9_{42}. [29]

Therefore, although surprisingly good at recognising chirality[30], the HOMFLY polynomial is not a complete invariant for this property.

2.2 Achirality of Composite Knots

A good way of confirming the chirality of a composite knot is by first finding its prime factors, and then calculating its HOMFLY polynomial using a unique property of the HOMFLY polynomial: $P(\mathbf{K}\#\mathbf{L}) = P(\mathbf{K}) \cdot P(\mathbf{L})$. If the product is not a self-conjugate polynomial, one can be certain that the composite knot is chiral.

One would expect the composition of two achiral knots to be achiral. In order to find out whether this is the case, let us consider the following conjecture:

A composite knot is achiral if all its factor knots are achiral. (*conjecture 1*)

Proof:

Let us use mathematical induction in order to prove *conjecture 1*. We will define $\mathbf{K}_n$ as the composition of *n* achiral factor knots. We shall assume that the conjecture is true for $n = s$. Therefore $\mathbf{K}_s$ is considered to be achiral.

If the conjecture is true, $\mathbf{K}_{s+1}$ should also be achiral. Due to the definition of composite knots, we know that $\mathbf{K}_{s+1} = \mathbf{K} \# \mathbf{K}_s$,where **K** is an achiral knot (see *figure 10*).

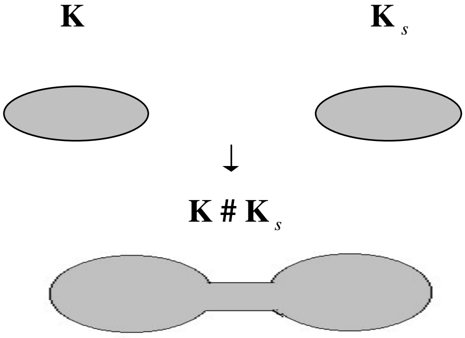

Figure 10: The composition $\mathbf{K} \# \mathbf{K}_s$.

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[28] Page 178 of [Adams].

[29] Picture obtained from Wolfram Research Inc, URL: http://mathworld.wolfram.com/knots/09-042.gif

[30] 9_{42} is the only chiral prime knot with a self-conjugate HOMFLY polynomial and less than 10 crossings.

In the composition $\mathbf{K} \# \mathbf{K}_s$**,** we are able to deform the crossings that were obtained from the factor knot **K** in such a way that this factor knot appears to be the mirror image $\mathbf{K}^*$ instead. Let us show that this can be done without changing those crossings that initially belonged to $\mathbf{K}_s$**.**

We will consider a composite knot to be what is called a *satellite* knot. In order to construct satellite knot, we first place a knot inside an unknotted solid torus in such a way that it hits each meridional disk of the torus[31] (and so that it cannot be deformed inside the torus to avoid this). Next, we knot the torus into the shape of a second knot, the *companion knot*. The knot inside the torus will then have the companion knot as its core curve (see *figure 11*). A composition of two knots **K** and **L** can now be generalised as a satellite knot with **K** placed inside a solid, knotted torus so that one of its strands reaches longitudinally around the torus. The shape of the torus will be that of the knot **L**.

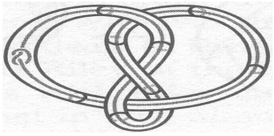

Figure 11: A *satellite* knot with a trefoil knot inside a solid torus, where the companion knot is 4_1 .[32]

Let us imagine $\mathbf{K} \# \mathbf{K}_s$ as a satellite knot for which **K** is the companion knot. Deforming the shape of the torus will not change those crossings that belong to the factor knot $\mathbf{K}_s$ inside the torus. Since **K** is achiral, we **can** deform the torus into the shape of the mirror image $\mathbf{K}^*$. This satellite knot, with $\mathbf{K}^*$ as its companion knot, is by definition the composite knot $\mathbf{K}_s \# \mathbf{K}^*$.

If we imagine $\mathbf{K}_s \# \mathbf{K}^*$ to be a satellite knot constructed in such a way that $\mathbf{K}_s$ is the companion knot, we **can** deform the achirally shaped torus into the shape of $\mathbf{K}_s^*$ without changing the crossings belonging to the factor knot $\mathbf{K}^*$ inside the torus. This satellite knot, with $\mathbf{K}_s^*$ as its companion knot, is by definition the composite knot $\mathbf{K}^* \# \mathbf{K}_s^*$**.** The knot $\mathbf{K}^* \# \mathbf{K}_s^*$, the composition of the mirror images of the original factor knots, must also be the mirror image of $\mathbf{K} \# \mathbf{K}_s$**.** Therefore, since $\mathbf{K}_{s+1} = \mathbf{K} \# \mathbf{K}_s$ and this is an ambient isotopy of its mirror image, it must be achiral.
Therefore if the conjecture is true for $n = s$, it must also be true for $n = s + 1$.

Let us prove the conjecture for $n = 2$, where the two factor knots are both prime knots 4_1.
First we will show that 4_1 is achiral by giving Reidemeister moves that deform a projection of 4_1 to its mirror image. Then $4_1 \# 4_1$ must also be achiral (see *figure 12).*

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[31] a meridional disk is a disk in a solid torus such that its boundary is a meridional curve, running once around the torus the "short" way.

[32] Picture from page 117 of [Adams]

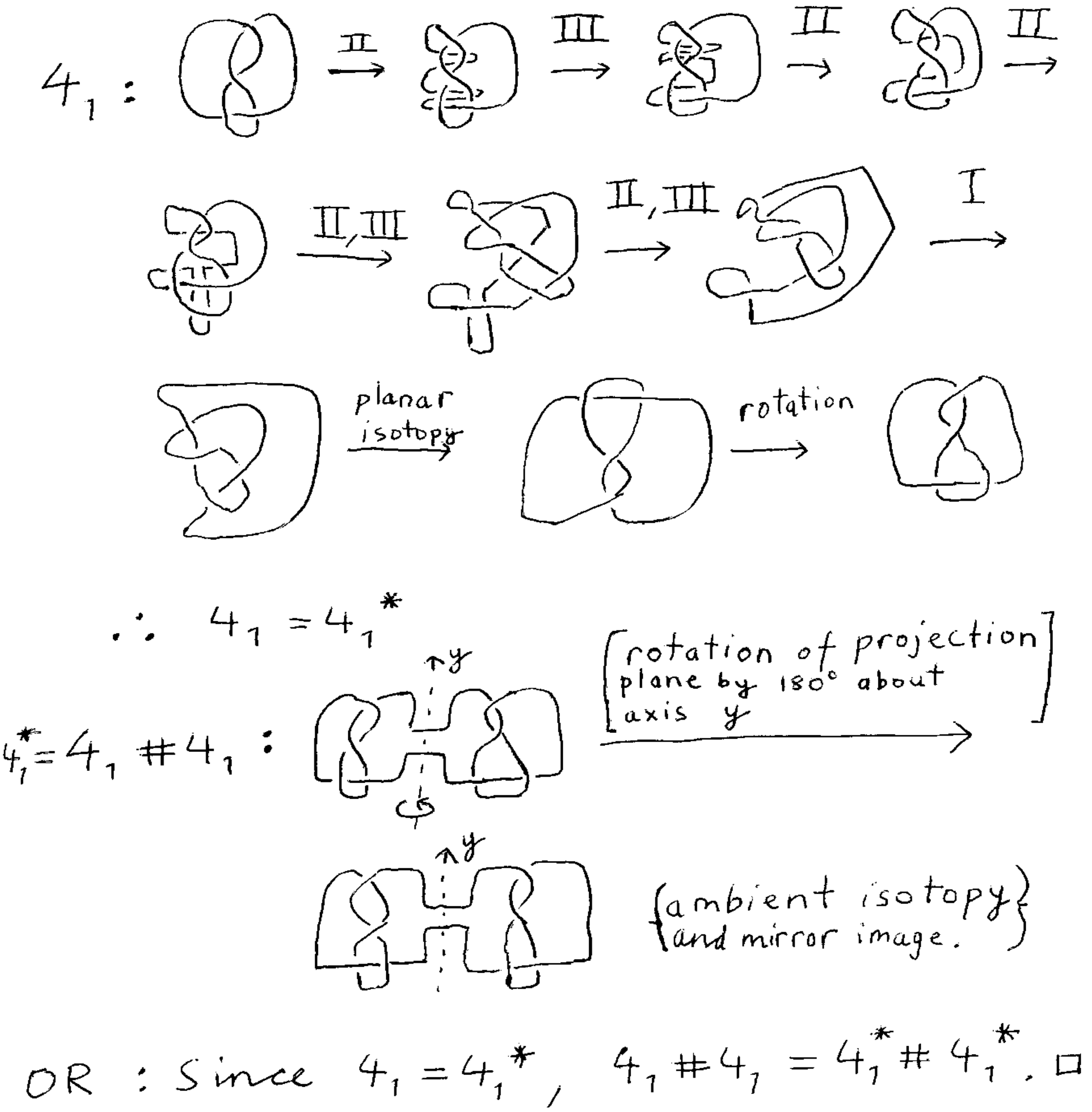

Figure 12: Reidemeister moves that prove the achirality of 4_1 and $4_1\#4_1$.

Therefore, since *conjecture 1* is true for $n = 2$, it must be true for all integers $n \geq 2$, and thus for all compositions of achiral knots. QED

Can compositions of chiral prime knots be achiral? One would expect the answer to be *no*, but for example the knot $3_1\#3_1^*$ (the composition of the left-hand and right-hand trefoil knots) is achiral[33]. We will generalise this result for all similar compositions in chapter 2.3.

2.3 Symmetry Presentations

Inspired by a conjecture about achiral knots by J.W. Gaberdiel (see *appendix 6.2*), an experiment was conducted by tying a 1.5 meter chord into the shapes of Rolfsen's[34] projections of achiral prime knots up to 8 crossings. It was attempted to deform the chord into shapes from which the achirality of the knots could easily be realised, and this resulted in the discovery of the following projections:

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[33] Professor Colin C. Adams kindly notified this in a brief e-mail discussion.

[34] Dale Rolfsen: *Knots and Links*, Publish or Perish Press, 1976.

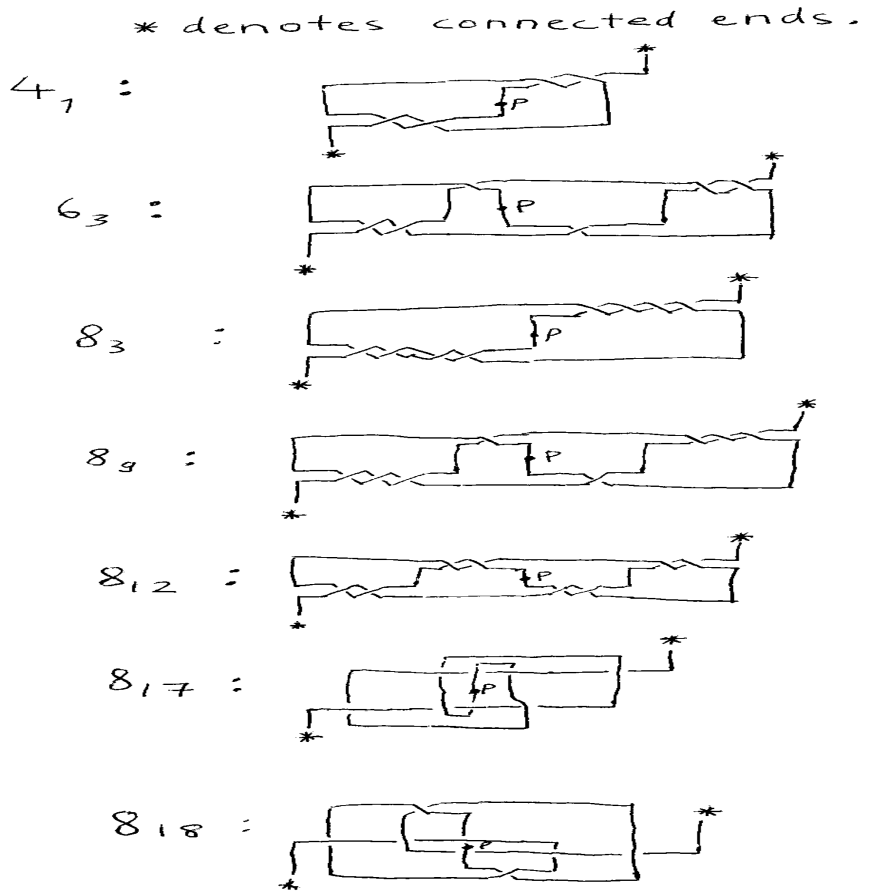

Figure 13: Centro-antisymmetrical projections of achiral prime knots through 8 crossings[35].

The term *rotation* is mathematically defined as a linear transformation represented by an *orthogonal matrix*[36] with determinant of value one. For the knots in *figure 13*, we can see that rotating the projection plane[37] by 180° about an axis passing through the centre of the knot projection and perpendicular to the projection plane yields the mirror image of the knot. The representations are said to have *2-fold*[38] *rotational antisymmetry* about this axis. It was observed that the projections of $4_1, 6_3, 8_3, 8_9$ and 8_{12} have certain similarities, which are generalised by defining a *sequential notation* for them. These knots are fully defined by a sequence of natural numbers $x_1, x_2, ..., x_n$ related to a general model for their centro-antisymmetrical projections (see *figure 14).* In this model we denote tangle x by $T(x)$ and tangle $-x$ by $T^*(x)$.

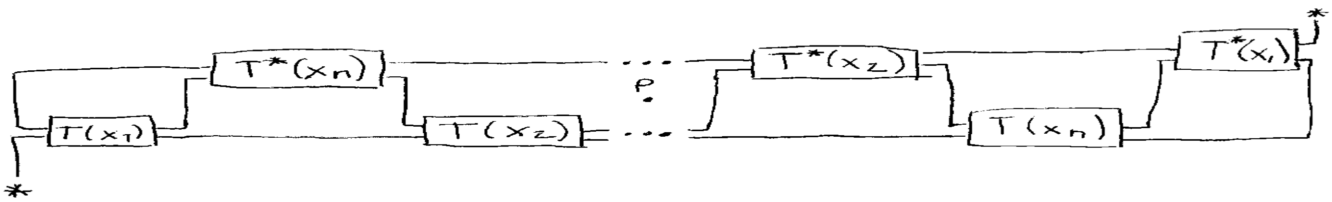

Figure 14: A general model for knots with a sequential notation.

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[35] Centro-antisymmetrical projections have rotational antisymmetry about an axis passing through the centre P of the figure. See the next page for more information about antisymmetry. Please note that these projections are assumed to be embedded in $\mathbf{S}^3$, so '*' – ends meet at ∞.

[36] A square matrix that is equal to the inverse of its transpose. [Nelson: *Penguin Dictionary of Mathematics*]

[37] The plane on which the knot is seen to 'lie".

[38] For n-fold rotational symmetry, the angle of rotation is $\frac{360^{\circ}}{n}$ [Nelson: *Penguin Dictionary of Mathematics*].

Further research shows that we have in fact redefined the Conway notation for achiral rational knots: the numbers in our notation correspond to half of the values in the *palindromic* Conway notation $a_1a_2...a_na_n...a_2a_1$ of an achiral rational knot (see *table 2*).

Alexander-Briggs notation	Sequential notation $\{x_m\}_{m=1}^n$	Conway notation
4_1	2	22
6_3	2,1	2112
8_3	4	44
8_9	3,1	3113
8_{12}	2,2	2222

Table 2: The sequential and Conway notations of achiral rational knots through 8 crossings

Since the sequential notation and the Conway notation of achiral rational knots are equivalent, we have found a centro-antisymmetrical projection for each achiral rational knot. The following is known for achiral rational knots:

A rational knot is achiral iff its Conway notation is palindromic and it has an even crossing number. All such knots have at least one projection with 2-fold rotational antisymmetry.

Theorem: Achiral rational knots[39].

In fact most achiral prime knots have projections with rotational antisymmetry. Usually these are realised from vertex-bicoloured graphs that are projected on either planes or spheres[40].

Twist knots are defined as prime knots with Conway notation *a*2, *a* > 0.[41] The composition of a twist knot and its mirror image is achiral, even in the case that the twist knot is chiral. This can be observed from *figure 15*, where *P* denotes the axis of 2-fold rotational antisymmetry.

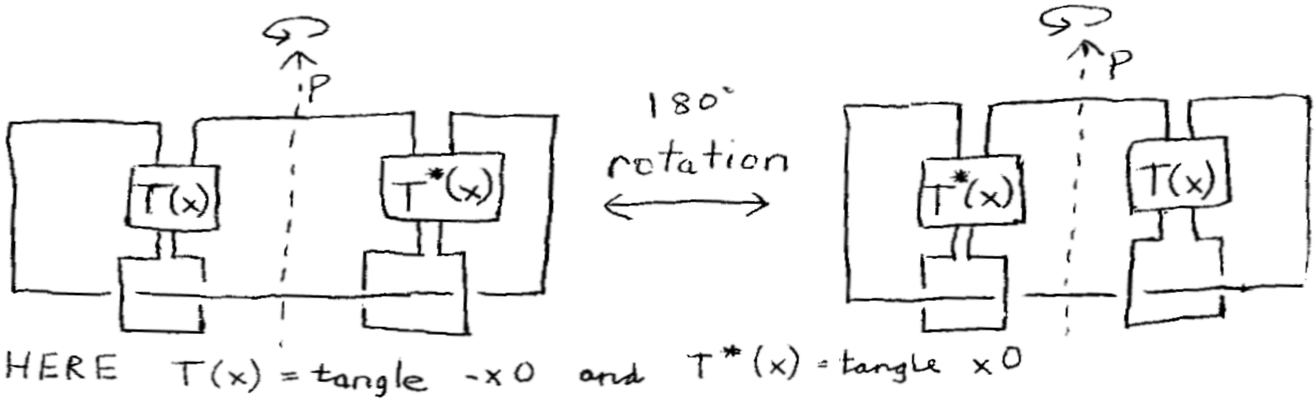

Figure 15: Achiral compositions of twist knots and their obverses.

2.4 Knot Determinants and Signatures

We would like to prove the chirality of those chiral knots that have self-conjugate HOMFLY polynomials. Within the prime knots of crossing number below 11, there are six knots whose chirality is

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[39] Slavik V. Jablan: *Ordering Knots*, URL: http://members.tripod.com/vismath/sl/link6.htm

[40] [Jablan], URL: http://members.tripod.com/vismath/sl/link16.htm

[41] Plus their mirror images.

not shown by their HOMFLY polynomials: $9_{42}, 10_{48}, 10_{71}, 10_{91}, 10_{104}$ and 10_{125}. In this chapter, we will consider methods that show the chirality of all, except 10_{48} and 10_{91}. The chirality of these two knots is shown by the two-variable *Kauffman polynomial F*, whose recurrence relation is slightly more complicated than that of the HOMFLY polynomial. The basic idea is the same: all achiral knots have self-conjugate Kauffman polynomials, but 10_{48} and 10_{91} do not – therefore they must be chiral. Although the Kauffman polynomial is more powerful than the HOMFLY polynomial[42], it is lengthier and takes longer to calculate. The Kauffman polynomial is not a complete invariant for achirality, since 9_{42} and 10_{125} have self-conjugate Kauffman polynomials[43].

For 58 years, the one-variable *Alexander polynomial* $\Delta_{\mathbf{K}}(t)$ was the only knot polynomial[44]. It is well-known that a knot and its mirror image always have the same Alexander polynomial. Contrarily to a common belief, the Alexander polynomial **does** introduce a means of detecting chirality: namely, the *determinant*[45] *value* of a knot given by $\Delta_{\mathbf{K}}(-1)$, where the polynomial is normalised so that $\Delta_{\mathbf{K}}(t) = \Delta_{\mathbf{K}}(t^{-1})$ and $\Delta_{\mathbf{K}}(1) = 1$. This normalised version of the Alexander polynomial is known as the *Conway-Alexander polynomial* $\nabla_{\mathbf{K}}(t)$.[46]

The information of the Conway-Alexander polynomial $\nabla_{\mathbf{K}}(t)$ is contained in the HOMFLY polynomial $P_{\mathbf{K}}(l,m)$ by the following relationship[47]:

$$\nabla_{\mathbf{K}}(t) = P_{\mathbf{K}}\left(i, i \cdot \left(t^{\frac{1}{2}} - t^{-\frac{1}{2}}\right)\right)$$

Therefore, the determinant det(**K**) of a knot **K** can be given by the HOMFLY polynomial:

$$\det(\mathbf{K}) = \nabla_{\mathbf{K}}(-1) = P_{\mathbf{K}}(i,-2)$$

Let us go through some of the *rules* stating how det(**K**) can determine achirality[48]:

1) If det(**K**) < 0, **K** is chiral (the signature $\sigma(\mathbf{K}) \equiv 2 \bmod 4 \rightarrow \sigma(\mathbf{K}) \neq 0$)[49].
2) If **K** is achiral and 3|det(**K**), then it must be that 9|det(**K**).
3) If **K** is achiral, det(**K**) mod 36 $\in \{1,5,9,13,17,25,29\}$. (This follows from *rule 4*)
4) An odd natural number *n* is the determinant of an achiral knot iff *n* is the sum of 2 squares[50].

---

[42] The Kauffman polynomial detects 99% of chiral knots with crossing number below 11. The figure is 97% for the HOMFLY.
[43] Alexander Stoimenow: *Sums of two squares and determinants of achiral knots*, University of Toronto, 2003, URL: http://arxiv.org/abs/math.GT/0003172
[44] Page 148 of [Adams]
[45] It is called a *determinant* because of its representation as a determinant of a Seifert matrix. A Seifert matrix describes a Seifert surface, that is an oriented topological surface with a knot as its boundary component.
[46] Eric W. Weisstein: *Alexander polynomial*, Wolfram Research Inc., URL: http://mathworld.wolfram.com/AlexanderPolynomial.html
[47] Page 174 of [Adams]
[48] For proofs of the statements, see reference [Stoimenow]
[49] See following page for information on signatures.

5) An odd natural number n is the determinant of an achiral rational knot iff n is the sum of the squares of 2 coprime[51] numbers. Due to Fermat's theorem[52] from 1640 concerning the sums of 2 squares, if $n = 4x + 1$ is a prime, it must be the determinant of a rational achiral knot.
6) An odd square n is the determinant of an achiral, alternating and prime knot iff $n \notin \{1,9,49\}$.

These are extraordinary rules. For example the chiral knot 9_{42} has the self-conjugate HOMFLY polynomial $P(9_{42}) = -2l^{-2} - 3 - 2l^2 + (l^{-2} + 4 + l^2)m^2 - m^4$. For det($9_{42}$), we have the following:

$$\boxed{\det(9_{42}) = P(i,-2) = -2i^{-2} - 3 - 2i^2 + \left(i^{-2} + 4 + i^2\right)\cdot(-2)^2 - (-2)^4 = -7}$$

Now, since $-7 < 0$, *rule 1* confirms that 9_{42} is chiral. This is also applicable to the knot 10_{125}.

Let us see Alexander Stoimenow's elegant proof for the *if*-part[53] of *rule 5* in order to show the connections between determinants and Conway notations of rational knots:

The Krebes' invariant of a tangle **T** can be defined by Kr(**T**) = $\frac{A}{B}$, where A is the determinant of the numerator closure of **T** and B is the determinant of the denominator closure of **T.**[54] An achiral rational knot (with palindromic Conway notation $a_1a_2...a_na_n...a_2a_1$) can be represented as the numerator closure of the sum of two tangles **T** and $\overline{\mathbf{T}}^T$, where $\overline{\mathbf{T}}^T$ is the transposition of the mirror image of **T** (see *figure 16*):

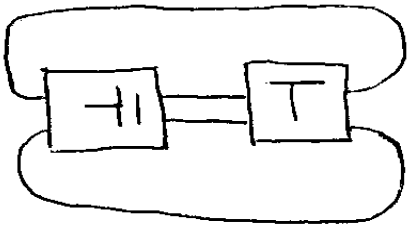

Figure 16: An achiral rational knot as the tangle sum $\mathbf{T}+\overline{\mathbf{T}}^T$

For rational tangles, the Krebes' invariant is exactly the continued fraction mentioned on *page 9*. The continued fraction for tangle **T** can be expressed in the form $\frac{p}{q}$, where p and q are coprime natural numbers and exactly one of them is odd. Since the Krebes' invariant is additive[55] under tangle sum and invertive under transposition, we have:

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[50] Please note that **all** knots have an odd determinant. Therefore, if a knot is achiral, its determinant is the sum of two squares. Unfortunately some chiral knots such as 10_{48} and 10_{91} have such a determinant too.

[51] Coprime numbers are not divisible by each other.

[52] And Euler's proof from 1754… See page 4 of [Stoimenow].

[53] That is excluding the *only if*–part. This proof was obtained from page 5-6 of [Stoimenow]

[54] See *page 7.*

[55] See *footnote 15.*on *page 6.*

$$\mathrm{Kr}(\mathbf{T}+\overline{\mathbf{T}}^{T})=\mathrm{Kr}(\mathbf{T})+\frac{1}{\mathrm{Kr}(\mathbf{T})}=\frac{p}{q}+\frac{q}{p}=\frac{p^2+q^2}{pq}=\frac{A}{B}$$

Now the determinant of the achiral rational knot is p^2+q^2, the sum of the squares of two coprime numbers. Furthermore, for two coprime natural numbers *p* and *q*, we can always represent $\frac{p}{q}$ as a continued fraction that corresponds to a rational tangle **T**. Then the odd natural number p^2+q^2 must be the determinant of an achiral rational knot $\mathbf{T}+\overline{\mathbf{T}}^{T}$. QED

The knot signature invariant $\sigma(\mathbf{K})$ of knot **K** is an even integer defined by the following skein relation[56]:

1) $\sigma(\mathbf{unknot})=0$.
2) $\sigma(\mathbf{L}_+)-\sigma(\mathbf{L}_-)\in\{0,2\}$.[57]
3) $4|\sigma(\mathbf{K})\leftrightarrow\nabla_{\mathbf{K}}(2i)>0$ (or $P_{\mathbf{K}}\left(i,\frac{i-3}{2}\right)>0$).

Achiral knots have $\sigma=0$. The signature can detect the chirality of 9_{42} and 10_{125}, which have $\sigma=-2$.

3. Conclusion and Evaluation

The HOMFLY polynomial is clearly among the most powerful invariants for chirality – even though it is slightly weaker than the Kauffman polynomial, it has several advantages:

1) The skein relation of the HOMFLY polynomial is among the simplest to memorise and use.
2) The property $P(\mathbf{K}\#\mathbf{L})=P(\mathbf{K})\cdot P(\mathbf{L})$ enables fast computation of polynomials of composite knots.
3) The HOMFLY polynomial contains information of the classical Alexander and Jones polynomials.

Let us look at which methods are sufficient enough to detect the chirality of the six prime knots ($9_{42}, 10_{48}, 10_{71}, 10_{91}, 10_{104}$ and 10_{125}) with crossing number below 11 and whose chirality the HOMFLY polynomial cannot detect:

- The chirality of 4 of the six knots, $9_{42}, 10_{71}, 10_{104}$ and 10_{125}, can be detected by the fact that their determinants are not the sums of two squares. Thus proving the chirality of many prime knots is extraordinarily easy due to the *rule 4* on *page 15.* In addition, the chirality of 10_{48} is proved by *rule 6*: it is clearly an alternating prime knot, but cannot be achiral, since $\det(10_{48})=49$.
- The signature can detect the chirality of 9_{42} and 10_{125}. It is clearly not as efficient as the determinant, but can still detect that 9_{42} is chiral, unlike the Kauffman polynomial.
- The Kauffman polynomial can detect the chirality of $10_{48}, 10_{91}, 10_{104}$ and 10_{125}. It is clearly a very powerful invariant for chirality and more efficient than the HOMFLY polynomial, although less user-friendly.

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[56] Eric W. Weisstein: *Knot Signature*, Wolfram Research Inc., URL: http://mathworld.wolfram.com/KnotSignature.html

[57] See *figure 7* on *page 8.*

In determining whether a given knot is achiral, it would be sensible to test the criteria mentioned in this essay in the following order:

1) Alternating knots with odd crossing number are chiral.
2) Composite knots with only achiral factors are achiral.
3) Rational knots are achiral only if they have palindromic Conway notations.
4) If the HOMFLY polynomial of a knot is not self-conjugate, the knot is chiral.
5) If the determinant of a knot is either negative or divisible by 3, but not 9, or if it is not the sum of two squares, the knot is chiral.
6) If the determinant of a prime, alternating knot is 1, 9 or 49, it is chiral.
7) If the signature invariant of a knot is not zero, the knot must be chiral.
8) If the Kauffman polynomial of a knot is not self-conjugate, the knot is chiral.

Finally, if one can find a knot projection that has rotational antisymmetry, one can be sure that the knot is achiral. We did not consider Milnor invariants or Vassiliev invariants of odd order[58], although they can also detect chirality[59].

Our consideration of achiral composite knots left an open question: Exactly which chiral factor knots yield achiral compositions (besides compositions of opposite-handed twist knots)?

Overall, there are a few efficient methods of determining whether a knot is achiral and these work most effectively when used complementarily with each other: The HOMFLY polynomial $P(l,m)$ together with its determinant value $P(i,-2)$ can be used to detect the chirality of all but one pair of chiral prime knots of crossing number below 11.[60] Whereas knot polynomials could be seen as steps towards a complete invariant of knots, that is distinguishing between **all** distinct knots, rotational antisymmetry addresses achirality exclusively: Knot polynomials might not be capable of detecting achirality perfectly until they can distinguish between all knots, while invariants specially designed for detecting achirality might succeed in this earlier. Finding an invariant that could definitively identify the chirality of any knot still remains one of the most difficult problems in knot theory – as the famous knot theorist Louis Kauffman put it: "A complete understanding of whether a knot is chiral still remains in the far distance."[61]

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[58] Jiang, Lin, Wang, Wu: *Achirality of Knots and Links*, Elsevier Science, 2002.

[59] These were beyond the scope of the essay.

[60] 10_{91} and its mirror image. The Kauffman polynomial can detect their chirality, but fails to do so for **two** other pairs.

[61] The essay Louis Kauffman: *Knots*, URL: http://www2.math.uic.edu/~kauffman/Tots/Knots.htm

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- Bogle, Ewing, Millet: *mkd2.exe* and *lmp1.exe*, URL: http://www.maths.warwick.ac.uk/~bjs/jones.html (Used for calculating HOMFLY, Jones, Alexander polynomials and determinants of self-drawn arbitrary knot projections.)

5. Acknowledgements

I would like to thank the following people for their replies via e-mail: [62]

- Professor Colin C. Adams, Williams College, Colin.Adams@williams.edu , confirmed that the compositions of achiral knots are achiral and noted that the composition of opposite-handed trefoil knots is achiral.
- Alexander Stoimenow, University of Toronto, stoimeno@math.toronto.edu , guided me to his paper on knot determinants (see bibliography).
- Justin William Gaberdiel, University of Arizona, eudidub@hotmail.com , discussed his ideas concerning self-entanglement with me, which led to the establishment of several new conjectures on self-entanglement and its connections with achirality.

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[62] The e-mail addresses of these people appear on their Internet sites: Therefore it is also seen as appropriate to include their e-mail addresses in this essay.

6.1 Knot Theory and Chirality: A Glossary[63]

achiral: describing equivalence with mirror image: an achiral *knot* is *ambient isotopic* to its mirror image, or its *obverse*.
achirality: similar to *chirality*, but the focus is on *achiral knots*. It is a property of *knots* with two values: 0 if the knot is *chiral*, 1 if the knot is *achiral*
alternating knot: a *knot* that has a *projection* which, when followed in one direction, alternates over and under itself at *crossings*.
ambient isotopy: a function that describes a continuous deformation of a topological object in $\mathbf{R}^3$ or the *three-sphere* $\mathbf{S}^3$.
amphicheiral: see *achiral* .
chemically chiral: a molecule is chemically chiral if the molecule can actually be deformed to its mirror image: therefore many molecules with the shape of *achiral knots* are still chemically chiral due to rigidness of bonds, among other things. A molecule with the shape of a *chiral knot* is always *chemically chiral*
chiral: the opposite of *achiral*, describing non-equivalence with mirror image.
chirality : a property of *knots* with two values: 1 if the knot is *chiral*, 0 if the knot is *achiral*.
companion knot: the outer shape of a *satellite knot*, the core curve of a knot *embedded* inside a solid torus. In other words, the companion knot is the shape of this solid torus.
composite knot: a *knot* formed by cutting arcs from the outer boundaries of two nontrivial (see *unknot*) knots and joining the ends to form one composite knot.
crossing: the point in a knot *projection* where a *knot* passes over (or conversely under) itself.
crossing change: see *obverse.*
crossing number: the minimum number of *crossings* in any of the *projections* of a knot.
embedding: a fixed placement of a topological object in $\mathbf{R}^3$ or the *three-sphere* $\mathbf{S}^3$. For graphs (sets of edges and vertices), an embedding is a particular way of connecting the same labelled vertices with edges.
factor knot: a factor knot is one of the knots used to form a *composite knot*.
invariant: a property or characteristic of a *knot* that always remains constant for the given knot. In practice, we check that deforming the knot does not change the characteristic by testing it with *Reidemeister moves*.
knot: a particular embedding of $\mathbf{S}^1$ in either $\mathbf{R}^3$ or $\mathbf{S}^3$, that is a simple continuous closed curve.
knot polynomial: an *invariant* of knots that is found by e.g. using a *skein relation*.
link: whereas a knot has only one closed component, a link may have more than one component (possibly intertwined).
orientation: one of two possible fixed directions assigned to a *projection* of a *knot*, usually denoted by arrows after each crossing.
obverse: the mirror image of a knot, i.e. a projection where each over*strand* of the original knot becomes an under*strand* of the obverse. This operation of switching over- and understrands is called a *crossing change*.
planar isotopy: A function that describes a continuous deformation of the strands in a knot projection, given that the crossings remain intact.
prime knot: a *knot* that is not a *composite knot*.
***projection*:** the visual two-dimensional representation of a knot, drawn as a line with gaps where the knot crosses under itself.
resolving tree: a diagram that shows how different *links* are related by a *skein relation*.
satellite knot: a knot formed by placing a nontrivial (see *unknot*) knot inside a solid torus and knotting the torus into the shape of the *companion knot*.
skein relation: A definition that determines how a *knot polynomial* changes if a small change is applied at one *crossing* of a *link projection*.
symmetry presentation: a projection of a knot that is *embedded* in such a way that it can simply be rotated to get its mirror image – we say that a knot is rigidly *achiral* if it has a symmetry presentation.
tangle: a fixed part of a knot projection with exactly four ends.
three-sphere: It can be defined as $\mathbf{S}^3 = \mathbf{R}^3 \cup \{\infty\}$.
***unknot*:** a knot that is *ambient isotopic* to a regular circle. Nontrivial knots are distinct from the trivial unknot.

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[63] Words that are written in italics have their own entry in the glossary.

6.2 Self-Entanglement

Self-entanglement is a concept introduced by J.W. Gaberdiel, a student who studied knot theory as a research project[64] at Arizona University in 2002. Self-entanglement (SE) is a construction method for knots and an attempt to describe "knottedness" [65] in a simplified manner, without constraining oneself to the visual illusion of knots "crossing" themselves in space.

Every knot can be constructed from the unknot by adding left- and right-handed *tentacles* to the knot. Each tentacle consists of a number of either left- or right-handed twists starting from a specified origin, attached to a specified target with a number of same-handed clasps (see *figure 11).*

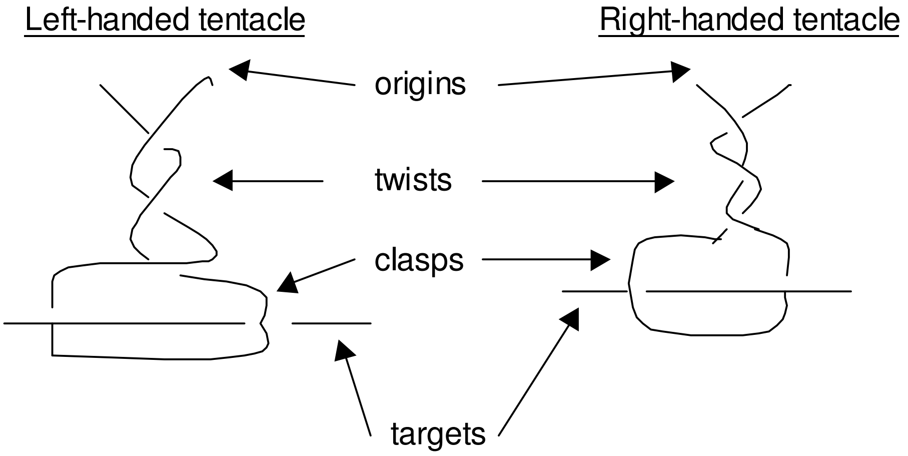

Figure : A 3-twist left-handed tentacle and a 3-twist right-handed tentacle.

Gaberdiel conjectured that achiral knots are exactly the knots that have two construction methods, otherwise similar, but with opposite handed twists and clasps. My own conjecture is that this works only for achiral rational knots: it is easy to prove the conjecture for achiral rational knots, but I have not yet been able to find two opposite-handed construction methods for any other achiral knots.

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[64] Justin William Gaberdiel: *A n Exploration of Nontraditional Conceptualizations of Knots*, University of Arizona, URL: http://hedgehog.math.arizona.edu/~ura/024/Dub.Gaberdiel/Knot_Report.pdf

[65] *Knottedness* is a term Gaberdiel uses to indicate the essential difference between distinct knots.